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THE VIBRATIONS OF AN EXTENSIBLE FLEXIBLE THREAD WITH A SMALL SAG†

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The free vibrations of an extensible flexible thread with a small sag are considered. A new form of solution of the equations of equilibrium of an extensible flexible thread is obtained. The differential equations of small vibrations about the equilibrium position are derived. An asymptotic analysis of the vibrations out of the vertical plane is carried out. It is established that these vibrations are close to the vibrations of a string. An asymptotic analysis of the low-frequency and high-frequency vibrations in the vertical plane is carried out. It is established that the natural frequencies and forms of the low-frequency vibrations depend very much on two small parameters: the parameter ε, characterizing the sag value, and the parameter δ, characterizing the degree of the thread stretching. It is proved that the low-frequency transverse vibrations when $\varepsilon^2/\delta \ll 1$ are close to the vibrations of a string, and when $\varepsilon^2/\delta \gg 1$ they are close to vibrations of an inextensible thread. If the quantities ε^2 and δ have the same asymptotic order, the most representative asymptotic form is obtained. In the high-frequency region there are long-wave longitudinal and short-wave transverse vibrations, irrespective of the ratio of the two small parameters. More complex forms of vibrations also sometimes arise, characterized by interaction between the longitudinal and transverse motions. Asymptotic expansions, which are uniformly valid over the whole frequency range, are obtained.

1. FORMULATION OF THE PROBLEM

We will consider an extensible flexible thread, the ends of which are clamped at the same height [1–3]. The position of a point on the thread is specified by the Lagrangian coordinate $\tilde{s}$—this is the length of the arc of the unstretched thread, measured from the lowest point of the thread in the equilibrium configuration. The origin of a Cartesian system of coordinates $\tilde{x}_1\tilde{x}_2\tilde{x}_3$ is placed at this point (Fig. 1).

The equation of motion of the thread has the form

$$\mu\frac{\partial^2\tilde{\mathbf{R}}}{\partial\tilde{t}^2}=\frac{\partial\tilde{\mathbf{Q}}}{\partial\tilde{s}}+\mathbf{q} \tag{1.1}$$

where $\tilde{\mathbf{R}}=\tilde{\mathbf{R}}(\tilde{s},\tilde{t})$ is the radius vector, defining the position of a point of the thread in space, $\tilde{t}$ is the time, μ is the mass per unit length of the unstretched thread, $\tilde{\mathbf{Q}}$ is the force over the cross-section of the thread, $\mathbf{q}=-q_0\mathbf{e}_2$, $q_0=\mu g$ is the external load acting on unit length of the unstretched thread.

The force in the flexible thread must be tangential to the curve specified by the radius vector $\tilde{\mathbf{R}}(\tilde{s},\tilde{t})$, which is expressed by the equation

$$\tilde{\mathbf{Q}}\times\frac{\partial\tilde{\mathbf{R}}}{\partial\tilde{s}}=\mathbf{0} \tag{1.2}$$

The value of the force over the cross-section of the thread is related to the relative elongation of the thread by Hooke's law

$$|\tilde{\mathbf{Q}}|=\tilde{c}\left(\left|\frac{\partial\tilde{\mathbf{R}}}{\partial\tilde{s}}\right|-1\right) \tag{1.3}$$

where c is the thread stiffness on stretching, defined in the simplest case by the formula $\tilde{c}=EF$, E is Young's modulus of the material and F is the cross-section area of the thread.

We will introduce the dimensionless variables and parameters

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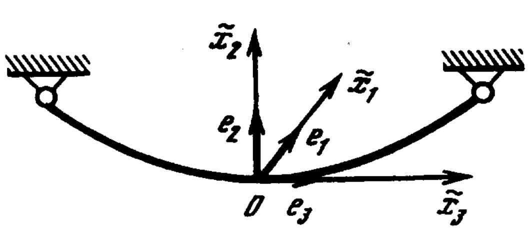

Fig. 1.

$$\mathbf{R}=\frac{\tilde{\mathbf{R}}}{\tilde{s}_0},\quad \mathbf{Q}=\frac{\tilde{\mathbf{Q}}}{H},\quad t=\Omega_0\tilde{t}\left(\Omega_0=\frac{1}{\tilde{s}_0}\sqrt{\frac{H}{\mu}}\right),\quad s=\frac{\tilde{s}}{\tilde{s}_0},\quad \varepsilon=\frac{q_0\tilde{s}_0}{H},\quad \delta=\frac{H}{\tilde{c}} \tag{1.4}$$

($\tilde{s}_0$ is the half-length of the unstretched thread and H is the tension in the thread at the lowest point of the equilibrium configuration).

Eliminating the force $\tilde{\mathbf{Q}}$ from Eqs (1.1)–(1.3) and using (1.4), we obtain the equation of motion of the extensible flexible thread in dimensionless variables (the dot denotes a derivative with respect to t and the prime denotes a derivative with respect to s)

$$\ddot{\mathbf{R}}=\frac{1}{\delta}\left((|\mathbf{R}'|-1)\frac{\mathbf{R}'}{|\mathbf{R}'|}\right)'-\varepsilon\mathbf{e}_2 \tag{1.5}$$

2. EQUILIBRIUM OF THE EXTENSIBLE FLEXIBLE THREAD

Integrating the differential equation of equilibrium, corresponding to (1.5) when $\ddot{\mathbf{R}}=\mathbf{0}$, we obtain the equilibrium configuration of the thread

$$\mathbf{R}_0=\left[\frac{1}{\varepsilon}\left(\sqrt{1+\varepsilon^2s^2}-1\right)+\delta\varepsilon\frac{s^2}{2}\right]\mathbf{e}_2+\left[\frac{1}{\varepsilon}\operatorname{arsh}\varepsilon s+\delta s\right]\mathbf{e}_3 \tag{2.1}$$

The unit vectors of the tangent, normal and binormal of curve (2.1) and their derivatives are given by the formulae

$$\mathbf{t}=\frac{\mathbf{R}_0'}{|\mathbf{R}_0'|}=\frac{\varepsilon s\mathbf{e}_2+\mathbf{e}_3}{\zeta},\quad \mathbf{n}=\frac{\mathbf{t}'}{|\mathbf{t}'|}=\frac{\mathbf{e}_2-\varepsilon s\mathbf{e}_3}{\zeta},\quad \mathbf{b}=\mathbf{t}\times\mathbf{n}=-\mathbf{e}_1$$
$$\mathbf{t}'=\frac{1}{\rho}\mathbf{n},\quad \mathbf{n}'=-\frac{1}{\rho}\mathbf{t},\quad \mathbf{b}'=\mathbf{0} \tag{2.2}$$

where $\zeta=\sqrt{(1+\varepsilon^2s^2)}$, $\rho=\zeta^2/\varepsilon$.

For an extensible flexible thread with a small sag, ε and δ are small parameters. The parameter ε represents the sag value; it can be shown that it is proportional to the ratio of the sag f to the length of the thread, $\varepsilon\approx 2f/\tilde{s}_0$. The parameter δ represents the degree of stretching of the thread and is equal to the relative elongation of the thread at the lowest point of the equilibrium configuration.

3. THE EQUATIONS OF SMALL VIBRATIONS

For small vibrations the radius vector of a point of the thread is given by the formula

$$\mathbf{R}=\mathbf{R}_0+\mathbf{u} \tag{3.1}$$

where $\mathbf{u}=\tilde{\mathbf{u}}/\tilde{s}_0$ is a small dimensionless displacement from the equilibrium configuration. Linearizing the equation of motion (1.5) in the region of the equilibrium configuration, we obtain the equation of small vibrations of an extensible flexible thread

$$\ddot{\mathbf{u}} = \frac{1}{\delta}\left(\frac{(|\mathbf{R}_0'|-1)}{|\mathbf{R}_0'|}\mathbf{u}' + \frac{(\mathbf{R}_0' \cdot \mathbf{u}')\mathbf{R}_0'}{|\mathbf{R}_0'|^3}\right)' \tag{3.2}$$

Specifying the vector of its displacement by the projections along the tangent, normal and binormal, $\mathbf{u} = w\mathbf{t} + v\mathbf{n} + u\mathbf{b}$, and substituting into (3.2) we obtain, taking (2.2) into account, the equations

$$\ddot{u} = \left(\frac{\zeta}{1+\delta\zeta}u'\right)'$$

$$\ddot{v} = \frac{1}{\delta\rho}\left(w' - \frac{v}{\rho}\right) + \left(\frac{\zeta}{1+\delta\zeta}\left(v' + \frac{w}{\rho}\right)\right)', \quad \ddot{w} = \frac{1}{\delta}\left(w' - \frac{v}{\rho}\right)' - \frac{\zeta}{\rho(1+\delta\zeta)}\left(v' + \frac{w}{\rho}\right) \tag{3.3}$$

the first of which describes vibrations out of the vertical plane, and the second and third describe vibrations in the vertical plane.

The boundary conditions have the form

$$s = \pm 1: \ u = v = w = 0 \tag{3.4}$$

Investigating the solutions of Eqs (3.3) in the form of principal vibrations

$$u = U(s)e^{i\omega t}, \quad v = V(s)e^{i\omega t}, \quad w = W(s)e^{i\omega t} \tag{3.5}$$

we obtain the equations

$$\omega^2 U + \left(\frac{\zeta}{1+\delta\zeta}U'\right)' = 0$$

$$\omega^2 V + \frac{\varepsilon}{\delta\zeta^2}\left(W' - \frac{\varepsilon}{\zeta^2}V\right) + \left(\frac{\zeta}{1+\delta\zeta}\left(V' + \frac{\varepsilon}{\zeta^2}W\right)\right)' = 0 \tag{3.6}$$

$$\omega^2 W + \frac{1}{\delta}\left(W' - \frac{\varepsilon}{\zeta^2}V\right)' - \frac{\varepsilon}{\zeta(1+\delta\zeta)}\left(V' + \frac{\varepsilon}{\zeta^2}W\right) = 0$$

4. VIBRATIONS OUT OF THE VERTICAL PLANE

Vibrations out of the vertical plane, described by the first equation of (3.6) are close to the vibrations of a string with a rectilinear axis and constant tension. Searching for an asymptotic expansion of the solution of this equation in the form

$$U = U_0 + O(\max(\delta, \varepsilon^2)), \quad \omega^2 = \omega_0^2(1 + O(\max(\delta, \varepsilon^2))) \tag{4.1}$$

we obtain for the principal term of the expansion a differential equation which defines the vibration form of a string. The principal term of the asymptotic expansion of the vibration frequencies is given by the formula

$$\omega_{0n} = n\pi / 2, \quad n = 1, 2, \ldots \tag{4.2}$$

which corresponds completely to the vibration frequencies of a string with a rectilinear axis of length $2s_0$, having constant tension H and a mass per unit length of μ. Note that the first frequency corresponds to "pendulum" vibrations of the thread. The vibration forms of the thread also correspond completely to the vibration forms of a string.

5. VIBRATIONS IN A VERTICAL PLANE

For vibrations in a vertical plane, described by the second and third equations of (3.6), we can distinguish low-frequency and high-frequency vibrations.

Low-frequency vibrations

A preliminary asymptotic analysis showed that the low-frequency vibration form depends very much on the ratio of the small parameters ε and δ. For a thread with a very small sag, and more accurately when $\varepsilon^2/\delta \ll 1$, the vibrations of an extensible thread are close to the vibrations of a string. For a slightly stretched thread, more accurately when $\varepsilon^2/\delta \gg 1$, the vibrations are close to those of an inextensible thread. We will carry out an asymptotic analysis for the most representative case, when the quantities ε^2 and δ have the same asymptotic order.

The solution of the second and third equations of (3.6) for low-frequency vibrations will be sought in the form

$$V = V_0 + O(\varepsilon), \quad W = \varepsilon W_1 + O(\varepsilon^2), \quad \omega^2 = \omega_0^2(1 + O(\varepsilon)) \tag{5.1}$$

For the principal terms of the asymptotic expansions we obtain the following system of differential equations and boundary conditions

$$V_0'' + \xi(W_1' - V_0) + \omega_0^2 V_0 = 0, \quad (W_1' - V_0)' = 0; \quad \xi = \varepsilon^2 / \delta$$
$$s = \pm 1: \quad V_0 = 0, \quad W_1 = 0$$

For symmetrical vibration forms (V_0 is an even function of s and W_1 is an odd function of s) the frequency equation has the form

$$(\xi / \omega_0^3)\sin\omega_0 + (1 - \xi / \omega_0^2)\cos\omega_0 = 0 \tag{5.2}$$

Note that for a very small sag ($\xi \ll 1$) the roots of the frequency equation

$$\omega_{0m} = \frac{2m-1}{2}\pi, \quad m = 1, 2, \ldots \tag{5.3}$$

give the frequencies of symmetrical vibrations, which are identical with the corresponding frequencies of a string with a rectilinear axis.

For a slightly stretched thread ($\xi \gg 1$) the frequency equation (5.2) can be written in the form

$$\sin\omega_0 - \omega_0\cos\omega_0 = 0$$

The natural frequencies can be determined, with sufficient accuracy, from the approximate formula

$$\omega_{0m} = \frac{(2m+1)\pi}{2}\left(1 - \left(\frac{2}{(2m+1)\pi}\right)^2\right), \quad m = 1, 2, \ldots \tag{5.4}$$

The main difference between the frequency spectrum (5.4) and (5.3) is the fact that for a slightly stretched thread there is no lowest frequency with number $m = 1$ from spectrum (5.3). This result was obtained previously in [1] for an inextensible thread. In the general case the natural frequencies of symmetrical vibrations are given by Eq. (5.2). A graph of the natural frequencies as a function of the ratio $\xi = \varepsilon^2/\delta$ is given in Fig. 2. As ξ increases the frequency with number m changes gradually and for large ξ has a value close to the frequency with number $m + 1$ for small ξ.

The symmetrical vibration forms are given by the formulae

$$\begin{aligned} V_m &= \cos\omega_{0m} - \cos\omega_{0m}s \\ W_m &= -(\varepsilon / \omega_{0m})[(\omega_{0m}^3 / \xi)\cos\omega_{0m}(1 - \xi / \omega_{0m}^2)s + \sin\omega_{0m}s] \end{aligned} \tag{5.5}$$

As the ratio $\xi = \varepsilon^2/\delta$ increases the vibration forms change so that the lowest vibration form with one half-wave in a span gradually changes into a form with three half-waves in a span, which is also the lowest form of symmetrical vibrations of a slightly stretched thread, that is shown on the right-hand side of Fig. 2. Note also that the string vibration form with three half-waves changes into the corresponding form with five half-waves, etc.

The frequency equation for the antisymmetric vibration forms (V_0 is an odd function of s and W_1 is an even function of s) has the form

$$\sin \omega_0 = 0 \tag{5.6}$$

The natural frequencies for the antisymmetric vibration forms are identical with the corresponding frequencies of a string with a rectilinear axis

$$\omega_{0n} = n\pi, \quad n = 1, 2, \ldots \tag{5.7}$$

while the vibration forms are given by the formulae

$$V_n = \sin \omega_{0n} s, \quad W_n = (\varepsilon / \omega_{0n})(\cos \omega_{0n} - \cos \omega_{0n} s) \tag{5.8}$$

Note that the vibration forms satisfy the orthogonality conditions

$$\int_{-1}^{1} V_{0r} V_{0p} ds = 0, \quad \int_{-1}^{1} [V'_{0r} V'_{0p} + \xi (W'_{1r} - V_{0r})(W'_{1p} - V_{0p})] ds = 0, \quad r \neq p$$

High-frequency vibrations

A preliminary analysis showed that the natural forms of high-frequency vibrations are the superposition of slowly changing functions $f_s(s_s)$, where $s_s = (1 + \varepsilon\delta_s + \ldots)s$ and rapidly changing oscillating functions $f_f(s_f)$, where $s_f = \delta^{-1/2}(1 + \varepsilon\delta_f + \ldots)s$. Here we have taken into account in explicit form the dependence of the arguments of the slowly and rapidly changing functions on the small parameters, as is done in the Linstedt–Poincaré method [4]. We have

$$df_s / ds = (1 + \varepsilon\delta_s + \ldots) f'_s, \quad df_f / ds = \delta^{-1/2}(1 + \varepsilon\delta_f + \ldots) f'_f$$
$$(f'_s = df_s / ds_s, \quad f'_f = df_f / ds_f)$$

The derivatives df_s/ds_s and df_f/ds_f are quantities of the order of unity. For slowly varying functions, their derivatives with respect to s and the functions themselves are of the same asymptotic order, while for the rapidly varying functions they have a different asymptotic order.

As a result of a preliminary analysis we established that the asymptotic expansion of the forms and frequencies of the vibrations have the form

$$V = \varepsilon V_s + \ldots + V_f + \ldots, \quad W = W_s + \ldots + \varepsilon\sqrt{\delta} W_f + \ldots$$

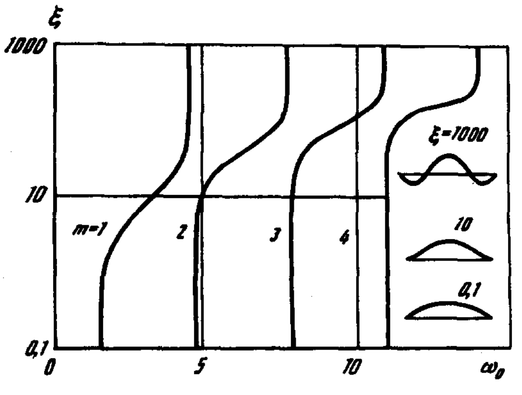

Fig. 2.

$$\omega^2 = \frac{\omega_0^2}{\delta}(1+\ldots), \quad \omega_0^2 = O(1) \tag{5.9}$$

Note that a preliminary analysis enables us to establish the relation between the asymptotic orders of the slowly varying components of the solution, and also the relation between the asymptotic orders of the rapidly varying components of the solution. The relation between the asymptotic orders of the slowly and rapidly varying functions will be established during the course of further constructions.

Substituting (5.9) into the last of the two equations in (3.6), noting that if the sum of the slowly and rapidly varying functions is equal to zero, then each of these functions is equal to zero, and equating the coefficients of the lowest asymptotic order of combinations of small parameters to zero, we obtain the following equations for the principal terms of the asymptotic expansions of the slowly varying functions

$$\omega_0^2 V_s + W_s' = 0, \quad W_s'' + \omega_0^2 W_s = 0 \tag{5.10}$$

and for the principal terms of the rapidly varying functions we obtain

$$V_f'' + \omega_0^2 V_f = 0, \quad -V_f' + W_f'' = 0 \tag{5.11}$$

Substituting (5.9) into the boundary conditions we obtain

$$s = \pm 1: \quad \varepsilon V_s + V_f = 0, \quad W_s + \varepsilon\sqrt{\delta} W_f = 0 \tag{5.12}$$

The frequency equation for symmetrical vibrations has the form

$$\sin\omega_0 \cos\frac{\omega_0}{\sqrt{\delta}} + O\left(\varepsilon^2\sqrt{\delta}\right) = 0 \tag{5.13}$$

For the vibration forms we have

$$\begin{aligned} V_m &= -\varepsilon\omega_{0m}\cos\frac{\omega_{0m}}{\sqrt{\delta}}\cos\omega_{0m}s + \varepsilon\omega_{0m}\cos\omega_{0m}\cos\frac{\omega_{0m}s}{\sqrt{\delta}} \\ W_m &= \omega_{0m}^2\cos\frac{\omega_{0m}}{\sqrt{\delta}}\sin\omega_{0m}s + \varepsilon^2\sqrt{\delta}\cos\omega_{0m}\sin\frac{\omega_{0m}s}{\sqrt{\delta}} \end{aligned} \tag{5.14}$$

where ω_{0m} is the mth root of frequency equation (5.13).

The frequency equation for the antisymmetric vibrations can be written in the form

$$\cos\omega_0 \sin\frac{\omega_0}{\sqrt{\delta}} + O\left(\varepsilon^2\sqrt{\delta}\right) = 0 \tag{5.15}$$

The vibration forms are given by the expressions

$$V_n = \cos\omega_{0n}\sin\frac{\omega_{0n}s}{\sqrt{\delta}}, \quad W_n = \frac{\varepsilon\sqrt{\delta}}{\omega_{0n}}\left(\cos\frac{\omega_{0n}}{\sqrt{\delta}}\cos\omega_{0n}s - \cos\omega_{0n}\cos\frac{\omega_{0n}s}{\sqrt{\delta}}\right) \tag{5.16}$$

where ω_{0n} is the nth root of frequency equation (5.15).

We will now analyse the results obtained. The frequency equation for the symmetrical vibrations (5.13) is satisfied when one or both factors on the left-hand side are small.

When the factor $\sin\omega_{0m}$, which is the predominant term governing the vibration form, is a small quantity, the term W_{sm} turns out to be slowly varying. Hence, in this case we obtain long-wave high-frequency longitudinal vibrations.

When the factor $\cos(\omega_{0m}/\sqrt{\delta})$, which is the predominant term, is a small quantity, the term V_{fm} will be rapidly varying. Hence, we obtain short-wave high-frequency transverse vibrations. Note that the frequency density of the transverse vibrations in the high-frequency band is considerably higher than the frequency density of longitudinal vibrations.

When both factors on the left-hand side of frequency equation (5.13) are small, we have a more complex pattern, determined by the interaction between the longitudinal and transverse vibrations.

An analysis of the frequency equation for antisymmetric vibrations (5.15) gives similar results.

Asymptotic expansions uniformly valid over the whole frequency band. We will obtain asymptotic expansions for the symmetrical and antisymmetrical vibrations, uniformly valid over the whole frequency band.

We will first consider symmetrical vibrations. The frequency equation for the low-frequency band (5.2) can be rewritten by replacing ω_0 by ω in the form

$$(\xi/\omega^3)\sin\omega+(1-\xi/\omega^2)\cos\omega=0 \tag{5.17}$$

The frequency equation for the high-frequency band (5.13) can be rewritten in the form

$$\left(\sin\omega\sqrt{\delta}/\left(\omega\sqrt{\delta}\right)\right)\cos\omega=0 \tag{5.18}$$

Note that Eqs (5.17) and (5.18) retain the same asymptotic accuracy as the initial equations (5.2) and (5.13). Equation (5.17) holds when $\omega = O(1)$, and Eq. (5.18) holds when $\omega = O(1/\sqrt{\delta})$.

Consider these equations in the intermediate region where

$$1<\operatorname{ord}\omega<1/\sqrt{\delta} \tag{5.19}$$

The corresponding formalism (see [5]) consists of the fact that we assume

$$\omega=\omega_\eta\sqrt{\eta/\delta} \tag{5.20}$$

where the quantity ω_η is fixed, while the asymptotic orders of the quantities ε, δ, η are such that when $\varepsilon \to 0$, $\delta \to 0$, $\eta \to 0$ the following inequalities are satisfied

$$\begin{aligned}&\operatorname{ord}\varepsilon^2<\operatorname{ord}\eta<1, \quad \text{or} \quad \varepsilon^2/\eta\to 0\\&\operatorname{ord}\delta<\operatorname{ord}\eta<1, \quad \text{or} \quad \eta/\delta\to\infty\end{aligned} \tag{5.21}$$

Substituting (5.20) into (5.17) and (5.18) and taking the limit (5.21), we obtain the following single equation in the intermediate region

$$\cos\omega = 0 \tag{5.22}$$

i.e. the low-frequency and high-frequency equations match. Moreover, we can construct an asymptotic frequency equation, uniformly valid over the whole frequency band, by adding the left-hand sides of (5.17) and (5.18) and subtracting the left-hand side of (5.22), which is valid in the intermediate region.

For the symmetrical vibrations the uniformly valid asymptotic frequency equation has the form

$$(\xi/\omega^3)\sin\omega+(\sin\omega\sqrt{\delta}/(\omega\sqrt{\delta})-\xi/\omega^2)\cos\omega=0 \tag{5.23}$$

By similar constructions we can obtain uniformly valid expressions for the symmetrical vibration forms

$$\begin{aligned}V_m&=\cos\omega_m\cos\omega_m\sqrt{\delta}s-\cos\omega_m\sqrt{\delta}\cos\omega_m s\\W_m&=-\frac{\varepsilon}{\omega_m}\left[\frac{\omega_m^3}{\xi}\cos\omega_m\left(\frac{\sin\omega_m\sqrt{\delta}s}{\omega_m\sqrt{\delta}}-\frac{\xi s}{\omega_m^2}\right)+\cos\omega_m\sqrt{\delta}\sin\omega_m s\right]\end{aligned} \tag{5.24}$$

Note that when $\omega_m = O(1)$, expressions (5.24) become expressions (5.5), and when $\omega_m = O(1/\sqrt{\delta})$ they become expressions (5.14) (apart from an unimportant normalizing factor).

Carrying out corresponding constructions for the antisymmetric vibrations, we obtain the following equations instead of (5.6) and (5.15)

$$\sin\omega=0, \quad \cos\omega\sqrt{\delta}\sin\omega=0 \tag{5.25}$$

which, in the intermediate region, convert into an equation identical with the first equation of (5.25).

For the antisymmetric vibrations the uniformly valid asymptotic frequency equation is identical with the second equation of (5.25), while the uniformly valid expressions for the vibration forms can be written in the form

$$V_n = \cos\omega_n\sqrt{\delta}\sin\omega_n s, \quad W_n = \frac{\varepsilon}{\omega}\left(\cos\omega_n\cos\omega_n\sqrt{\delta}s - \cos\omega_n\sqrt{\delta}\cos\omega_n s\right) \tag{5.26}$$

When $\omega_n = O(1)$ expressions (5.26) become expressions (5.8), and when $\omega_n = O(1/\sqrt{\delta})$ they become expressions (5.16).

We note in conclusion that a similar method of asymptotic analysis of the low-frequency and high-frequency vibrations and the construction of asymptotic frequency expansions that are uniformly valid over the whole frequency band can also be employed when investigating more complex systems, such as, for example, rods, plates and shells.

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